The Contextual Character of Modal Interpretations of Quantum Mechanics

Graciela Domenech[*1,3], Hector Freytes [2] and Christian de Ronde [3,4]

1. Instituto de Astronomía y Física del Espacio (IAFE)
Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina
2. Dipartimento di Scienze e Pedagogiche e Filosofiche - Universita degli Studi di Cagliari
Via Is Mirrionis 1, 09123, Cagliari - Italia
3. Center Leo Apostel (CLEA)
4. Foundations of the Exact Sciences (FUND)
Brussels Free University - Krijgskundestraat 33, 1160 Brussels - Belgium

Abstract

In this article we discuss the contextual character of quantum mechanics in the framework of modal interpretations. We investigate its historical origin and relate contemporary modal interpretations to those proposed by M. Born and W. Heisenberg. We present then a general characterization of what we consider to be a modal interpretation. Following previous papers in which we have introduced modalities in the Kochen-Specker theorem, we investigate the consequences of these theorems in relation to the modal interpretations of quantum mechanics.

Introduction

Modal propositions refer to possibility and necessity, about what 'must be' and what 'may be' the case. If one thinks about how things are in the actual world, then one may also think of how things might have been in an alternative, non actual but possible, state of affairs. Modalities were introduced in physics by statistical mechanics in order to take into account possible configurations of physical systems regardless of what is actually the case. In contemporary physics, the development of quantum theories has thrown new light to the problem of the relation between possible and actual. We intend to analyze the rôle of modalities in quantum mechanics, precisely in what sense we may say that a system possibly possesses a property. We also study whether the addition of modal propositions to the lattice of actual propositions would circumvent the contextual character of the quantum discourse about properties.

The paper is organized in five sections. In section 1, we study the modal character of early interpretations of quantum mechanics. In section 2, we discuss the contextual character of the theory. In section 3, we outline a brief review of contemporary modal interpretations. We then present, in section 4, a general characterization of these interpretations. In section 5 we analyze their contextual character. Finally, we present a brief discussion.

1 Early Modal Interpretations of Quantum Mechanics

The first consistent formalization of the methods applied to the study of quantum phenomena was developed in the form of 'matrix mechanics' first under the ideas of W. Heisenberg, and then in collaboration with M. Born and P. Jordan, during the second half of 1925 (Heisenberg, 1925; Born and Jordan, 1925; Born, Heisenberg and Jordan, 1926):

[*] Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)

> "[...] in the summer of 1925 it led to a mathematical formalism called matrix mechanics or, more generally, quantum mechanics. The equations of motion of Newtonian mechanics were replaced by similar equations between matrices; it was a strange experience to find that many of the old results of Newtonian mechanics, like conservation of energy, etc, could be derived also in the new scheme. Later the investigations of Born, Jordan and Dirac showed that the matrices representing position and momentum of the electron did not commute. This latter fact demonstrated clearly the essential difference between quantum mechanics and classical mechanics." W. Heisenber (1958, p. 41)

In January 1926 E. Schrödinger completed the first part of his apparently quite different approach to the same problem in terms of eigenfunctions of a differential equation, which is known as 'wave mechanics' (Schrödinger, 1926a). Schrödinger had read Einstein's paper on the quantum theory of the ideal gas. He recognized later it was these ideas, due to Einstein and de Broglie, which allowed him to develop his own theory (Jammer, 1966, p. 257). By March 1926, Schrödinger interpreted the wave function ψ as a 'mechanical field scalar' (*mechanischer Feldskalar*) in connection with the classical theory of electromagnetic radiation. Following this idea, the eigenvalues where taken as frequency values (Schrödinger, 1926c). The quantum postulate was then conceived in connection with resonance phenomena, making possible to reject the existence of discrete energy levels and quantum jumps. Schrödinger's wave theory was well received by the orthodox physics community which wanted to put an end to the deviations of the atomic theory from the classical conception of natural sciences developed from Newton to Einstein. At the time Heisenberg was quite unhappy about this turn:

> "In July [1926] I visited my parents in Münich and on this occasion I heard a lecture given by Schrödinger for the physicists in Münich about his work on wave mechanics. It was thus that I first became acquainted with the interpretation Schrödinger wanted to give his mathematical formalism of wave mechanics, and I was disturbed about the confusion with which I believed this would burden atomic theory. Unfortunately, nothing came from my attempt during the discussion to put things in order. My argument that one could not even understand Plank's radiation law on the basis of Schrödinger's interpretation convinced no one. Wilhem Wien, who held the chair of experimental physics at the University of Münich, answered rather sharply that one must put an end to quantum jumps and the whole atomic mysticism, and the difficulties I had mentioned would certainly soon be solved by Schrödinger." W. Heisenberg (quoted from Wheeler and Zurek, 1983, p. 56)

Schrödinger related the matter field and the electromagnetic emission or absorption of atomic systems in the famous paper in which he showed the equivalence between matrix and wave mechanics (Schrödinger, 1926b). To do so, he first linked the wave function ψ to a space density of the electrical charge given by the real part of $\psi \partial \psi^* / \partial t$, where ψ^* denotes the complex conjugate of ψ, correcting it later to $\psi \psi^*$. This interpretation, in terms of a field, had many difficult features and internal inconsistencies, which would only be solved, later on, by M. Born.

1.1 Born's Modal Interpretation of the Quantum Wave Function

At the same time of Schrödinger's demonstration of the equivalence of matrix and wave mechanics, Born presented his probabilistic interpretation of the wave function. As commented by M. Jammer:

> "Almost simultaneously with the appearance of Schrödinger's fourth communication, a new interpretation of the ψ-function was published that had far-reaching consequences for modern physics not only from the purely technical point of view but also with respect to the philosophical significance of its content. Only four days after Schrödinger's concluding contribution had been sent to the editor of the *Annalen der Physik* the publishers of the *Zeitschrift für Phisik* received a paper, less than five pages long, titled 'On the Quantum Mechanics of Collision Processes' in which Max Born proposed, for the first time, a probabilistic interpretation of the wave function implying thereby that microphysics must be considered a probabilistic theory." M. Jammer (1974, p. 38)

In his original article, Born begins by explicitly characterizing quantum mechanics as a modal theory, emphasizing its probabilistic character but stressing at the same time the indeterministic element present in the theory:

> "Schrödinger's quantum mechanics [therefore] gives quite a definite answer to the question of the effect of the collision; but there is no question of any causal description. One gets no answer to the question, 'what is the state after the collision' but only to the question, 'how probable is a specified outcome of the collision'.
>
> Here the whole problem of determinism comes up. *From the standpoint of our quantum mechanics there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision.* [...] I myself am inclined to give up determinism in the world of the atoms. But that is a philosophical question for which physical arguments alone are not decisive." M. Born (1926, quoted from Wheeler and Zurek, 1983, p. 57, emphasis added)

It soon became evident that the interpretation of modality in the new theory departed from its use in classical statistical mechanics as lack of knowledge. In the years to come this would remain one of the main themes regarding the physical interpretation of quantum mechanics.

> "It was wave or quantum mechanics that was first able to assert *the existence of primary probabilities in the laws of nature*, which accordingly do not admit of reduction to deterministic natural laws by auxiliary hypotheses, as do for example the thermodynamic probabilities of classical physics. This revolutionary consequence is regarded as irrevocable by the great majority of modern theoretical physicists -primarily by M. Born, W. Heisenberg and N. Bohr, with whom I also associate myself." W. Pauli (1994, p. 46)

1.2 Heisenberg's Modal Interpretation

Even though Born's interpretation of the wave function might seem to solve the interpretational problems by making reference to probabilities, it became immediately clear that this resource would not avoid the main difficulties. As Schrödinger noted in a letter to Einstein dated November 18, 1950, the so called 'quantum probabilities' departed from what was commonly understood as *probability*:

> "It seems to me that the concept of probability is terribly mishandled these days. Probability surely has as its substance a statement as to whether something *is* or *is not* the case –an uncertain statement, to be sure. But nevertheless it has meaning only if one is indeed convinced that the something in question quite definitely *is* or *is not* the case. A probabilistic assertion presupposes the full reality of its subject." E. Schrödinger (1950, quoted from Bub, 1997, p. 115)

Heisenberg himself was eager to repeatedly remark the essential distance between classical and quantum probabilities:

> "For each complementary statement the question [whether the atom is in the left or the right half of the box] is not decided. But the term 'not decided' is by no means equivalent to the term 'not known'. 'Not known' would mean that the atom is 'really' left or right, only we do not know where it is. But 'not decided' indicates a different situation, expressible only by a complementary statement." W. Heisenberg (1958, p. 158).

His own interpretation, although not completely clear, related possibility to the Aristotelian concept of *potentia*:

> "[...] the paper of Bohr, Kramers and Slater revealed one essential feature of the correct interpretation of quantum theory. This concept of the probability wave was something entirely new in theoretical physics since Newton. Probability in mathematics or in statistical mechanics means a statement about our degree of knowledge of the actual situation. In throwing dice we do not know the fine details of the motion of our hands which determine the fall of the dice and therefore we say that the probability for throwing a special number is just one in six. The probability wave function of Bohr, Kramers and Slater, however, meant more than that; it meant a tendency for something. It was a quantitative version of the old concept of 'potentia' in Aristotelian philosophy. It introduced something standing in the middle between the idea of an event ant the actual event, a strange kind of physical reality just in the middle between possibility and reality." W. Heisenberg (1958, p. 42)

It is important to remark that in Heisenberg's interpretation the projection postulate is not considered to be a physical process as is the case, for example, of GRW-type theories which intend to provide a dynamical account of the collapse of the wave function (Ghirardi, Rimini, Weber, 1985):

> "The observation itself changes the probability function discontinuously; it selects of all possible events the actual one that has taken place. [...] When the old adage Natura non facit saltus is used to criticism of quantum theory, we can reply that certainly our knowledge can change suddenly and that this fact justifies the use of the term quantum jump." W. Heisenberg (1958, p. 54)

Modal interpretations describe the evolution of possibilities. However, this representation must not contradict a particular actualization, one must be able to give an account of the path from the possible to the actual.

> "The mathematical representation of the physical process changes discontinuously with each new measurement; the observation singles out of a large number of possibilities one of which is the one which has happened. The wave packet which has spread out is replaced by a smaller one which represents the result of this observation. As our knowledge of the system does change discontinuously at each observation its mathematical representation must also change discontinuously; this is to be found in classical statistical theories as well as in the present theory." W. Heisenberg (1949, p. 36)

From the beginning it was acknowledged that quantum theory started with a paradox: on the one hand, it describes experiments in terms of classical physics and, on the other, it was born from the very critical distance taken from exactly these same concepts (the description via the trajectory). The necessity of a new image, an *anschaulich* content, was early recognized by Heisenberg, but in order to provide such visualization one needed to understand the structure of the theory. Classically, systems are described in terms of their actual properties, thus, in order to create a new image consistent with the quantum formalism, it appeared important to study the new structure of propositions about the properties of the system. This task was accomplished by the developments of G. Birkhoff and J. von Neumann, in what they called quantum logic. However, this analysis did not include modal features. The analysis of the modal properties was always placed within the limits of Bohr's concept of complementarity; thus, the classical space-time description (given by the definite result, i.e. a spot in a photographic plate) was complementary to the quantum causal description (provided by the Schrödinger evolution of the quantum wave function) in a quite definite sense:

> "[...] as a geometric or kinematic description of a process implies observation, it follows that such a description of atomic processes necessarily precludes the exact validity of the law of causality–and conversely. Bohr has pointed out that it is therefore impossible to demand that both requirements be fulfilled by the quantum theory. They represent complementary and mutually exclusive aspects of atomic phenomena." W. Heisenberg (1949, pp. 63-64)

The relation between these descriptions, namely, the study of the relation between the possible and the actual is the main subject of this work. In the following section we will come to the concept of contextuality, which rigorously summarizes the ideas of Bohr and Heisenberg as discussed above.

2 The Contextual Character of Quantum Mechanics

As we have already mentioned, when discussing the interpretation of matrix mechanics, Heisenberg could not avoid recognizing that his formalism did not provide a consistent image (an *anschaulich* content), contrary to wave mechanics which could be interpreted in terms of a field. As noted by J. Hilgevoord and J. Uffink (2006), the purpose of his 1927 paper was to provide exactly this lacking feature. In this paper, he developed the undeterminancy relations, which in turn present one of the most striking features of quantum systems, namely the fact that exact values cannot be simultaneously assigned to all their quantities. This characteristic of the new mechanics, as well as its consequences, was clearly acknowledged by Schrödinger:

> "[...] if I wish to ascribe to the model at each moment a definite (merely not exactly known to me) state, or (which is the same) to all determining parts definite (merely not exactly known to me)

numerical values, then there is no supposition as to these numerical values to be imagined that would not conflict with some portion of quantum theoretical assertions." E. Schrödinger (1935, quoted from Wheeler and Zurek, 1983, p. 152, emphasis added)

Also P.A.M. Dirac stated in his famous book:

> "The expression that an observable 'has a particular value' for a particular state is permissible in quantum mechanics in the special case when a measurement of the observable is certain to lead to the particular value, so that the state is an eigenstate of the observable. It may easily be verified from the algebra that, with this restricted meaning for an observable 'having a value', if two observables have values for a particular state, then for this state the sum of the two observables (if the sum is an observable) has a value equal to the sum of the values of the two observables separately and the product of the two observables (if this product is an observable) has a value equal to the product of the values of the two observables separately." P. Dirac (1958, p. 46).

This last point is the requirement of the *functional compatibility condition* (FUNC), to which we will return later.

In the usual terms of quantum logic (see for example Jauch, 1968; Piron, 1976), a property of a system is related to a subspace of the Hilbert space of its (pure) states or, analogously, to the projector operator onto that subspace. A physical magnitude $\mathcal{A}$ is represented by an operator $\mathbf{A}$ acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition $\mathbf{A} = \sum_i a_i \mathbf{P}_i = \sum_i a_i |a_i >< a_i|$ are satisfied. The real numbers a_i are related to the outcomes of measurements of the magnitude $\mathcal{A}$ and projectors $|a_i >< a_i|$ to the properties of the physical system. Precisely, let $\mathcal{H}$ be the Hilbert space associated to the physical system and $\mathcal{L}(\mathcal{H})$ be the set of closed subspaces on $\mathcal{H}$. If we consider the set of these subspaces ordered by inclusion, with the complement defined by orthocomplementation and meet and join operations defined by intersection and direct sum of subspaces, then $\mathcal{L}(\mathcal{H})$ is a complete orthomodular lattice (Maeda and Maeda, 1970). It is well known that each self-adjoint operator $\mathbf{A}$ has associated a Boolean sublattice W_A of $\mathcal{L}(\mathcal{H})$. More precisely, W_A is the Boolean algebra of projectors $\mathbf{P}_i$ of the spectral decomposition. We will refer to W_A as the spectral algebra of the operator $\mathbf{A}$. Any proposition about the system is represented by an element of $\mathcal{L}(\mathcal{H})$, which is the algebra of quantum logic introduced by Birkhoff and von Neumann (1936).

A complete set of properties of a system that may be simultaneously predicated allows to construct a Boolean propositional system. This is usually referred to as a *context*. In terms of operators, it is in correspondence with a complete set of commuting observables, CSCO for short. Assigning values (a set of their corresponding eigenvalues) to these magnitudes poses no difficulties. But if we try to interpret eigenvalues as the actual values of the physical properties of a system, we are faced to all kind of no-go theorems that preclude this possibility. Most remarkably is the Kochen-Specker (KS) theorem that rules out the non-contextual assignment of values to physical magnitudes (Kochen and Specker, 1967). An explicit statement of the KS theorem reads (Held, 2003):

Theorem 2.1 *Let $\mathcal{H}$ be a Hilbert space of dimension greater than 2 of the states of the system and M be a set of observables, represented by operators on $\mathcal{H}$. Then, the following two assumptions are contradictory:*

1. *All members of M simultaneously have values, i.e. are unambiguously mapped onto real unique numbers (designated, for observables $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ... by $v(\mathbf{A})$, $v(\mathbf{B})$, $v(\mathbf{C})$, ...).*

2. *Values of observables conform to the following constrains:*

   If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all compatible and $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then $v(\mathbf{C}) = v(\mathbf{A}) + v(\mathbf{B})$;

   if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all compatible and $\mathbf{C} = \mathbf{AB}$, then $v(\mathbf{C}) = v(\mathbf{A})v(\mathbf{B})$. □

As we have stated in (Domenech and Freytes, 2005), KS theorem may be expressed in terms of families of Boolean homomorphisms. Assigning values to a physical quantity $\mathcal{A}$ is equivalent to establishing a Boolean homomorphism $v : W_A \to \mathbf{2}$ (Isham and Butterfield, 1998), being $\mathbf{2}$ the two elements Boolean algebra. Thus, we may define a *global valuation* over $\mathcal{L}(\mathcal{H})$ as the family of Boolean homomorphisms $(v_i : W_i \to \mathbf{2})_{i \in I}$ such that $v_i \mid W_i \cap W_j = v_j \mid W_i \cap W_j$ for each $i, j \in I$, being $(W_i)_{i \in I}$ the family of Boolean sublattices of $\mathcal{L}(\mathcal{H})$. This global valuation would give the values of all magnitudes at the same time maintaining a *compatibility condition* in the sense that whenever two magnitudes shear one or more

projectors, the values assigned to those projectors are the same from every context. But KS theorem assures that we cannot assign real numbers pertaining to their spectra to the operators in such a way to satisfy *item* 2 of 2.1, i.e., FUNC, the expression of the 'natural' requirement asked by Dirac, as we have mentioned above. KS theorem expresses the fact that, if we demand a valuation to satisfy FUNC, then it is forbidden to define it in a non-contextual fashion for subsets of quantities represented by commuting operators. In the algebraic terms of the previous definition, KS theorem reads:

Theorem 2.2 *If $\mathcal{H}$ is a Hilbert space such that $dim(\mathcal{H}) > 2$, then a global valuation over $\mathcal{L}(\mathcal{H})$ is not possible.* □

Of course contextual valuations allow us to refer to different sets of actual properties of the system which define its state in each case. Algebraically, a *contextual valuation* is a Boolean valuation over one chosen spectral algebra. In classical particle mechanics it is possible to define a Boolean valuation of all propositions, that is to say, it is possible to give a value to all the properties in such a way of satisfying FUNC. This possibility is lost in the quantum case and it is not a matter of interpretation, it is the underlying mathematical structure that enables this possibility for classical mechanics and forbids it in the quantum case. In fact, one may also arrive at contextuality from a topological analysis (Domenech and Freytes, 2005). To show it, let us briefly recall that if I is a topological space, a *sheaf* over I is a pair (A, p) where A is a topological space and $p : A \to I$ is a local homeomorphism. This means that each $a \in A$ has an open set G_a in A that is mapped homeomorphically by p onto $p(G_a) = \{p(x) : x \in G_a\}$, and the latter is open in I. It is clear that p is continuous and open map. *Local sections* of the sheaf p are continuous maps $\nu : U \to I$ defined over open proper subsets U of I such that the following diagram is commutative:

$$\begin{tikzcd} U \arrow[r, "\nu"] \arrow[dr, "1_U"'] \arrow[dr, phantom, shift left=3ex, "\equiv"] & A \arrow[d, "p"] \\ & U \end{tikzcd}$$

In particular we use the term *global section* only when $U = I$.

Thus, we may consider the family $\mathcal{W}$ of all Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$ ordered by inclusion and the topological space $\langle \mathcal{W}, \mathcal{W}^+ \rangle$. On the set $E = \{(W, f) : W \in \mathcal{W},\ f : W \to \mathbf{2}$ *is a Boolean homomorphism*$\}$ we define a partial ordering given as $(W_1, f_1) \leq (W_2, f_2) \iff W_1 \subseteq W_2$ *and* $f_1 = f_2 \mid W_1$. Thus we consider the topological space $\langle E, E^+ \rangle$ whose canonical basis is given by the principal decreasing sets $((W, f)] = \{(G, f \mid G) : G \subseteq W\}$. By simplicity $((W, f)]$ is noted as $(W, f]$. Then, the map $p : E \to \mathcal{W}$ such that $(W, f) \mapsto W$ is a sheaf over $\mathcal{W}$. We refer to it as the *spectral sheaf*. We have shown elsewhere (Domenech and Freytes, 2005) that, if $\nu : U \to E$ is a local section of the spectral sheaf p, then for each $W \in U$ we have that $\nu(W) = (W, f)$ for some Boolean homomorphism $f : W \to \mathbf{2}$ and also, if $W_0 \subseteq W$, then $\nu(W_0) = (W_0, f \mid W_0)$. From the physical perspective, we may say that the spectral sheaf takes into account the whole set of possible ways of assigning truth values to the propositions associated with the projectors of the spectral decomposition $\mathbf{A} = \sum_i a_i \mathbf{P}_i$.

The continuity of a local section of p guarantees that the truth value of a proposition is maintained when considering the inclusion of subalgebras. In this way, the *compatibility condition* of the Boolean valuation with respect of intersection of pairs of Boolean sublattices of $\mathcal{L}(\mathcal{H})$ is maintained. A global section $\tau : \mathcal{W} \to E$ of p is interpreted as follows: the map assigns to every $W \in \mathcal{W}$ a fixed Boolean valuation $\tau_w : W \to \mathbf{2}$ obviously satisfying the compatibility condition. So KS theorem in terms of the spectral sheaf reads:

Theorem 2.3 *If $\mathcal{H}$ is a Hilbert space such that $dim(\mathcal{H}) > 2$ then the spectral sheaf p has no global sections.* □

We may build a *contextual valuation* in terms of a local section as follows: Let A be a physical magnitude with known value, i.e. we have been able to establish a Boolean valuation $f : W_A \to 2$. It is not very hard to see that the assignment $\nu : (W] \to E$ such that for each $W_i \in (W]$, $\nu(W_i) = (W_i, f \mid W_i)$ is a local section of p. To extend contextual valuations we turn now to consider local sections. To do this, let ν be a local section of p and W_A the spectral algebra associated to the operator $\mathbf{A}$. Then an extended

valuation over A is given by the set $\bar{\nu}(A) = \{W_B \in dom(\nu) : W_B \subseteq W_A\}$ and it is easy to prove that, if ν is a local section of p and W_A the spectral algebra associated to the operator $\mathbf{A}$, then: $\bar{\nu}(A)$ is a decreasing set and, if $W_A \in U$, then $\bar{\nu}(A) = (W_A]$. Valuations are deeply connected to the election of particular local sections of the spectral sheaf. So we see here once more that we cannot speak of the value of a physical magnitude without specifying this election, that clearly means the election of a particular context.

3 Contemporary Modal Interpretations of Quantum Mechanics

Modal interpretations of quantum mechanics intend to provide an objective reading of the mathematical formalism "in terms of properties possessed by physical systems, independently of consciousness and measurements (in the sense of human interventions)" (Dieks, 2005). Contemporary versions of the modal approach were faced with the problem of doing so, staying away from contradictions delivered by the KS theorem. B. van Fraassen was the first one to formally include the reasoning of modal logic in quantum mechanics. He presented a modal interpretation of quantum logic in terms of its semantical analysis (van Fraassen 1973, 1981). This analysis had the purpose to clarify which properties among those of the complete set structured in the lattice of subspaces of Hilbert space pertain to the system:

> "The interpretational question facing us is exactly: in general, which value attributions are true? The response to this question can be very conservative or very liberal. Both court later puzzles. I take it that the Copenhagen interpretation -really, a roughly correlated set of attitudes expressed by members of the Copenhagen school, and not a precise interpretation- introduced great conservatism in this respect. Copenhagen scientists appeared to doubt or deny that observables even have values, unless their state forces to say so. I shall accordingly refer to the following very cautious answer as the *Copenhagen variant* of the modal interpretation. It is the variant I prefer." B. van Fraassen (1991, p. 280).

After van Fraassen's interpretation, S. Kochen presented his own modal version, also a continuation of the early founding fathers discussions. C. F. von Weizsäcker refers to it in the following terms:

> "We consider it is an illuminating clarification of the mathematical structure of the theory, especially apt to describe the measuring process. We would, however feel that it means not an alternative but a continuation to the Copenhagen interpretation (Bohr and, to some extent, Heisenberg)." Th. Görnitz and C. F. von Weizsäcker (1987, p. 357)

In this section we intend to discuss the different versions of the modal interpretation in order to delineate a general characterization. We concentrate on van Fraassen's Copenhagen variant, Kochen-Dieks modal interpretation, Bub's Bohmian variant and the atomic interpretation of Bacciagaluppi and Dickson, because they comprise the main features of all modal interpretations. (See also Healey 1989, Vermaas and Dieks 1995, Bene and Dieks 2002; and Vermaas 1999a, Dickson 2002, Dieks and Vermaas 1998 and de Ronde 2003 for general reviews) In particular, we want to analyze the interpretational rules which account for the path from the possible to the actual, and the meaning of modality. For the sake of simplicity, we will be concerned here only with pure states. This can be extended to density operators as shown by Vermaas and Dieks (1995).

3.1 Van Fraassen's Copenhagen Variant

As noted by Dirac in the first chapter of his famous book, the existence of superpositions is responsible of the striking difference of quantum behaviour from classical one. In fact, the photon being in a superposition of translational states must be accepted if we want to explain interference effects (Dirac, 1958). Superpositions are also central when dealing with the measurement process, where the various terms associated with the possible outcomes of a measurement must be assumed to be present together in the description. This conduces to the so called "measurement problem". Thus, the individual occurrence of a measurement outcome is out of the scope of quantum mechanics, and only the probability of such result is given by the state of the system via the Born rule. This fact leads van Fraassen to the distinction between *value-attributing propositions* and *state-attributing propositions*, between *value-states* and *dynamic-states*:

"[...] a *state*, which is in the scope of quantum mechanics, gives us only probabilities for actual occurrence of *events* which are outside that scope. They can't be entirely outside the scope, since the events are surely described if they are assigned probabilities; but at least they are not the same things as the states which assign the probability.

In other words, the state delimits what can and cannot occur, and how likely it is -it delimits possibility, impossibility, and probability of occurrence- but does not say what actually occurs." B. van Fraassen (1991, p. 279)

So van Fraassen distinguishes propositions about events and propositions about states. Propositions about events are *value-attributing propositions* $< \mathbf{A}, \sigma >$, they say that 'Observable $\mathbf{A}$ has certain value belonging to a set σ'. Propositions about states are of the form 'The system is in a state of this or that type' (in a pure state, in some mixture of pure states, in a state such that...). A *state-attribution proposition* $[\mathbf{A}, \sigma]$ gives a probability of the value-attribution proposition, it states that a measurement of $\mathbf{A}$ must have an outcome in σ. *Value-states* are specified by stating which observables have values and which they are. *Dynamic-states*, by stating how the system will develop if isolated, and how if acted upon in any definite, given fashion. This is endowed with the following interpretation:

"The interpretation says that, if a system X has dynamic state φ at t, then the state-attributions $[\mathbf{A}, \sigma]$ which are true are those that $Tr(\rho \mathbf{P}^{\mathbf{A}}_{\sigma}) = 1$. [$\mathbf{P}^{\mathbf{A}}_{\sigma}$ is the projector over the corresponding subspace.] About the value-attributions, it says that they cannot be deduced from the dynamic state, but are constrained in three ways :

1. If $[\mathbf{A}, \sigma]$ is true then so is the value-attribution $< \mathbf{A}, \sigma >$: observable $\mathbf{A}$ has value in σ.
2. All the true value-attributions could have Born probability 1 together.
3. The set of true value-attributions is maximal with respect to the feature (2.)" B. van Fraassen (1991, p. 281).

This distinction between value-attribution propositions and state-attribution propositions allows van Fraassen to face the measurement problem from a new position. The way out of the contradiction between the presence of various results associated to the different terms in a superposition and the appearance of only one result proposed by von Neumann is the projection postulate, the non-causal state transition. In his spirit, an observable pertaining to a system has a value if and only if the system is in a corresponding eigenstate of the observable (the eigenstate-eigenvalue link). So, the observable, say $\mathbf{A}$, has a value if and only if a measurement of $\mathbf{A}$ is certain to have certain outcome. If the outcome of the measurement is uncertain, which is the case when the state is in a superposition of eigenstates of the observable, then the observable has no value.

Van Fraassen proposes to emphasize this modal character of the theory via the rôle of the state:

"[...] the transition from the possible to the actual is not a transition *of* state, but a transition *described by* the state." B. van Fraassen (1991, p. 279)

and to interpret the emergence of a result in a new light:

"[...] [the emergence of a result is] *as if the Projection Postulate were correct.* For at the end of a measurement of $\mathbf{A}$ on system X, it is indeed true that $\mathbf{A}$ has the actual value which is the measurement outcome. But, of course, the Projection Postulate is not really correct: there has been a transition from possible to actual value, so what it entailed about values of observables is correct, but that is all. There has been no acausal state transition." B. van Fraassen (1991, p. 288)

We now briefly recall the main aspects of van Fraassen's modal interpretation in terms of quantum logic, following (van Fraassen 1991, chap. 9). We concentrate on those aspects that will be necessary in order to compare with our own treatment of modal propositions (see section 5). In the modal interpretation, the probabilities are of events, each describable as 'an observable having a certain value', which are aspects of the value states. If w is a physical situation in which system X exists, then X has both a *dynamic state* φ and a *value state* λ, i.e. $w =< \varphi, \lambda >$. A *value state* λ is a map of observable $\mathbf{A}$ into non-empty Borel sets σ such that it assigns $\{1\}$ to $1_\sigma \mathbf{A}$. 1_σ is the characteristic function of a set σ of values. So, if the observable $1_\sigma \mathbf{A}$ has value 1, then it is impossible that $\mathbf{A}$ has a value outside σ. The proposition $< \mathbf{A}, \sigma >= \{w : \lambda(w)(\mathbf{A}) \subseteq \sigma\}$ assigns values to physical magnitudes, it is a

value-attribution proposition and is read as '**A** (actually) has value in σ'. $\mathcal{V}$ is called the set of value attributions $\mathcal{V} = \{< \mathbf{A}, \sigma >$: **A** an observable and σ a Borel set$\}$. The logic operations among value-attribution propositions are defined as: $< \mathbf{A}, \sigma >^{\perp} = < \mathbf{A}, \Re - \sigma >$, $< \mathbf{A}, \sigma > \wedge < \mathbf{A}, \theta > = < \mathbf{A}, \sigma \cap \theta >$, $< \mathbf{A}, \sigma > \vee < \mathbf{A}, \theta > = < \mathbf{A}, \sigma \cup \theta >$ and $\wedge\{< \mathbf{A}, \sigma_i >: i \in \mathcal{N}\} = < \mathbf{A}, \cap\{\sigma_i : i \in \mathcal{N}\} >$. With all this, $\mathcal{V}$ is the union of a family of Boolean sigma algebras $< \mathbf{A} >$ with common unit and zero equal to $< \mathbf{A}, S(\mathbf{A}) >$ and $< \mathbf{A}, \wedge >$ respectively. The Law of Excluded Middle is satisfied: every situation w belongs to $q \vee q^{\perp}$, but not the Law of Bivalence: situation w may belong neither to q nor to $q^{\perp}$.

A *dynamic state* φ is a function from $\mathcal{V}$ into [0, 1], whose restriction to each Boolean sigma algebra $< \mathbf{A} >$ is a probability measure. The relation between dynamic and value state is the following: φ and λ are a dynamic state and a value state respectively, only if there exist possible situations w and w' such that $\varphi = \varphi(w)$, $\lambda = \lambda(w')$. Here, φ is an eigenstate of **A**, with corresponding eigenvalue a, exactly if $\varphi(< \mathbf{A}, \{a\} >) = 1$. The *state-attribution proposition* $[\mathbf{A}, \sigma]$ is defined as: $[\mathbf{A}, \sigma] = \{w : \varphi(w)(< \mathbf{A}, \sigma >) = 1\}$ and means '**A** must have value in σ'. $\mathcal{P}$ denotes the set of state-attribution propositions: $\mathcal{P} = \{[\mathbf{A}, \sigma] : \mathbf{A}$ an observable, σ a Borel set$\}$. Partial order between them is given by $[\mathbf{A}, \sigma] \subseteq [\mathbf{A}', \sigma']$ only if, for all dynamical states φ, $\varphi(< \mathbf{A}, \sigma >) \leq \varphi(< \mathbf{A}', \sigma' >)$ and the logic operations are (well) defined as: $[\mathbf{A}, \sigma]^{\perp} = [\mathbf{A}, \Re - \sigma]$, $[\mathbf{A}, \sigma] \uplus [\mathbf{A}, \theta] = [\mathbf{A}, \sigma \cup \theta]$ and $[\mathbf{A}, \sigma] \cap [\mathbf{A}, \theta] = [\mathbf{A}, \sigma \cap \theta]$. With all this, $< \mathcal{P}, \subseteq, ^{\perp} >$ is an orthoposet, the orthoposet formed by 'pasting together' a family of Boolean algebras in which whole operations coincide in areas of overlap. It may be enriched to approach the lattice of subspaces of Hilbert space.

One may recognize a modal relation between both kind of propositions. One starts denying the collapse in the measurement process and recognizing that the observable has one of the possible eigenvalues. Then it may be asked what may be inferred with respect of those values when one knows the dynamic state. The answer van Fraassen gives is that, in the case that $\varphi(w)$ is an eigenstate of the observable **A** with eigenvalue a, then **A** actually does have value a. This means that in this case, the measurement 'reveals' the value the observable already had. He generalizes this idea and postulates that $[\mathbf{A}, \sigma]$ implies $< \mathbf{A}, \sigma >$. With this assumption and the rejection of an ignorance interpretation of the uncertainty principle, he is able to prove that $[\mathbf{A}, \sigma] = \Box < \mathbf{A}, \sigma >$. $\Box$ is defined by $\Box Q = \{w :$ for all w', if wRw' then $w' \in Q\}$, where q is any proposition and R the relative possibility relation: w' is possible relative to w exactly if, for all Q in $\mathcal{V}$, if w is in Q then w' is in Q. So, $[\mathbf{A}, \sigma]$ may be read as 'necessarily, $< \mathbf{A}, \sigma >$'. This says that the dynamic state assigns 1 to $< \mathbf{A}, \sigma >$ if and only if the value state that accompanies any relatively possible dynamic state makes $< \mathbf{A}, \sigma >$ true. Instead of the transitive possibility relation R, one may use an equivalence relation to define the necessity operator $\breve{\Box}$. In this case, van Fraassen maintains that the map $[\mathbf{A}, \sigma] \rightarrow < \mathbf{A}, \sigma >$ is an isomorphism of posets $< \mathcal{P}, \subseteq >$ and $< \mathcal{V}, \subseteq >$ and, when orthocomplementation if defined, it becomes an isomorphism between the orthoposets. Thus, the logic of $\mathcal{V}$ is that of $\mathcal{P}$, i.e., quantum logic. Endowed with these tools, van Fraassen gives an interpretation of the probabilities of the measurement outcomes which is in agreement with the Born rule. To do this, he considers mixtures. However, in spite of his motivation, we will continue restricting to the case of pure states because our aim is to analyze structures which gather together possibility and actuality. In fact, our approach allows us to organize a propositional system that takes into account the 'paste' of propositions about actual properties and modal properties. (Domenech *et al.*, 2006a)

3.2 Kochen-Dieks Modal Interpretation

The next modal interpretation we would like to review is due to S. Kochen and D. Dieks (K-D, for short), who proposed to use the so called biorthogonal decomposition theorem (also called Schmidt theorem) in order to describe the correlations between the quantum system and the apparatus in the measurement process:

Theorem 3.1 *Given a state $|\Psi_{\alpha\beta}\rangle$ in $\mathcal{H} = \mathcal{H}_\alpha \otimes \mathcal{H}_\beta$. The Schmidt theorem assures there always exist orthonormal bases for $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$, $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ such that $|\Psi_{\alpha\beta}\rangle$ can be written as:*

$$|\Psi_{\alpha\beta}\rangle = \sum c_j |a_j\rangle \otimes |b_j\rangle.$$

The different values in $\{|c_j|^2\}$ represent a spectrum of the Schmidt decomposition given by $\{\lambda_j\}$. Every λ_j represents a projection in $\mathcal{H}_\alpha$ and a projection in $\mathcal{H}_\beta$ defined as $P_\alpha(\lambda_j) = \sum |a_j\rangle \langle a_j|$ and $P_\beta(\lambda_j) = \sum |b_j\rangle \langle b_j|$, respectively. Furthermore, if the $\{|c_j|^2\}$ are non degenerate, there is a one-to-one correlation between the projections $P_\beta(\lambda_j)$ and $P_\alpha(\lambda_j)$ pertaining to subsystems $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ given by each value of the spectrum λ_j. $\Box$

Through this theorem one is able to distinguish, by tracing over the degrees of freedom of the subspace $\mathcal{H}_\alpha$ or the subspace $\mathcal{H}_\beta$, between system and apparatus (for a proof of the theorem see Bacciagaluppi 1996, section 2.3). As noted by Kochen (1985, p. 152): "Every interaction gives rise to a unique correlation between certain canonically defined properties of the two interacting systems. These properties form a Boolean algebra and so obey the laws of classical logic." The biorthogonal decomposition gives in this way, a one to one relation between the apparatus and the quantum system and the following interpretation: *The system α possibly possesses one of the properties $\{|a_j\rangle\ \langle a_j|\}$, and the actual possessed property $|a_k\rangle\ \langle a_k|$ is determined by the observation that the device possesses the reading $|b_k\rangle\ \langle b_k|$.* However, by tracing over the degrees of freedom of the system, one obtains an *imporper mixture*. It is well known that improper mixtures cannot be interpreted in terms of ignorance (D'Espagnat, 1976), and thus, one comes back to the problem of interpreting modalities. Following van Fraassen's distinction between value states and dynamical states, Dieks solves the problem of putting together the seemingly incompatible character of improper mixtures and ignorance via the distinction between different levels of discourse. For example, with respect to Schrödinger's cat, Dieks (1988a, p. 189) states that: "It is the state vector which is in a superposition, not the cat itself. 'State vector' and 'cat' are two concepts at different levels of discourse." In order to explicitly take into account this remark, P. Vermaas and D. Dieks distinguish between *physical properties* and *mathematical properties*:

> "[...] the interpretation gives the probabilities of various possible states of affairs of which it is assumed that only one is actually realized. [...] it is crucial to notice the distinction which is made in the modal interpretation between physical properties and mathematical states. The latter are defined in Hilbert space and encode probabilistic information about the properties (values of physical magnitudes) which are present in the system itself. There is no one-to-one relation between physical properties and mathematical states. Now, the ignorance inherent in the modal interpretation pertains to physical properties; not to mathematical states." P. Veramaas and D. Dieks (1995, p. 155)

Once again, the main problem remains that of interpreting possibility and its relation to actuality:

> "[...] an irreducible statistical theory only speaks about possible outcomes, not about the actual one; this predicts only probability distributions of all outcomes, and says nothing about the result which really will be realized in a single case. In brief, such a theory is not about what is real and actual but only about what could be the case." D. Dieks (1988a, p. 177)

Kochen (1985, p. 152) and Dieks do not consider the collapse of the state vector as a physical process: "[...] the collapse of the wave function is not a real physical effect in this interpretation but is simply the result of a change in perspective from our witnessing system to another." However, they must give an account of the emergence of a single result, they thus appeal to an interpretational rule in an analogous fashion to van Fraassen's proposal:

> "I now propose the following interpretational rule: as soon as there is a unique decomposition of the form (2), the partial system represented by the $|\phi_k\rangle$, taken by itself, can be *described* as possessing one of the values of the physical quantity corresponding to the set $|\phi_k\rangle$, with probability $|c_k|^2$. [By Eq (2), they mean $|\Psi_{\alpha\beta}\rangle = \sum c_j|\phi_j\rangle\ \otimes |R_j\rangle$]
>
> This rule is intended to have the following important consequence. Experimental data that pertain only to the object system, and that say it possesses the property associated with, e.g., $|\phi_1\rangle$, not only count as support for the theoretical description $|\phi_1\rangle|R_1\rangle$ but also as empirical support for the theoretical description (2)." D. Dieks (1988b, p. 39)

It is important to notice that the seemingly *ad hoc* move of using a preferred basis (such as the Schmidt basis) can be given a physical motivation. It has been proved by Dieks (1995) that, given the following two conditions:

1. one-to-one correlation: we require a one to one correlation between the definite properties of the system and the definite properties of its environment,
2. no hidden variables: the Hilbert space formalism, with the usual representation of physical magnitudes by observables, should be completely respected,

the only basis that accomplishes these two conditions is the Schmidt basis. The first demand appears as obvious when reflecting on the preconditions which allow us to talk about measurement. The second demand can be considered as a commitment to the early interpretation of Bohr, Born, Heisenberg and Pauli; to consider the quantum description as providing all there is to know with respect to atomic events, and that one should not seek for a more complete causal description in terms of space and time. As expressed by Bohr (1928), and also by Heisenberg (1949, section 3), these two descriptions are complementary to each other (see also Dieks, 1989, section 9).

It is important to stress at this point that, given the complete system and its corresponding Hilbert space, the choice of what is the system under study and what is the apparatus determines the factorization of the complete space that leads to the set of definite properties given by the Schmidt decomposition. In spite of the fact that this cut is (mathematically) not fixed *a priori* in the formalism, the (physical) choice of the apparatus determines explicitly the context. It is exactly this possibility, of having different incompatible contexts given by the choice of mutually incompatible apparatuses, which in turn determines KS type contradictions within the modal interpretation (see, for a discussion, Karakostas 2004, section 6.1).

As noted by Dieks in a recent paper (Dieks, 2007): "According to the modal interpretation the state in Hilbert space thus is about possibilities, about what may be the case; about modalities. But there is also a second aspect to ψ: it is the theoretical quantity that occurs in the evolution equation, and its evolution governs deterministically how the set of definite valued quantities changes. This double role of ψ, on the one hand probabilistic and on the other dynamical and deterministic, is a well-known feature of the Bohm interpretation." We will now turn our attention to Bub's Bohmian variant in order to discuss further the relation between Bohmian mechanics and the modal interpretation.

3.3 Bub's Bohmian Variant

The modal version of J. Bub recalls on D. Bohm's interpretation and proposes to take some observable, $\mathbf{R}$, as *always* possessing a definite value. In this way one can avoid KS contradictions and maintain a consistent discourse about statements which pertain to the sub-lattice determined by the *preferred observable* $\mathbf{R}$. As van Fraassen's and Vermaas and Dieks' interpretations, Bub's proposal distinguishes between *dynamical states* and *property* or *value states*, in his case with the purpose of interpreting the wave function as defining a Kolmogorovian probability measure over a restricted subalgebra of the lattice $\mathcal{L}(\mathcal{H})$ of projection operations (corresponding to yes-no experiments) over the state space. It is this distinction between property states and dynamical states that gives a modal character to the interpretation:

> "The idea behind a 'modal' interpretation of quantum mechanics is that quantum states, unlike classical states, constrain possibilities rather than actualities -which leaves open the question of whether one can introduce property states [...] that attribute values to (some) observables of the theory, or equivalently, truth values to the corresponding propositions." J. Bub (1997, p. 173)

In precise terms, as $\mathcal{L}(\mathcal{H})$ does not admit a global family of compatible valuations, and thus not all propositions about the system are determinately true or false, probabilities defined by the (pure) state cannot be interpreted epistemically (Bub, 1997, 119). But, if one chooses, for a given state $|e>$, a "preferred observable" $\mathbf{R}$, these properties can be taken as determinate since the propositions associated with $\mathbf{R}$, i.e., with the projectors in which $\mathbf{R}$ decomposes, generate a Boolean algebra. Bub constructs the maximal sublattices $\mathcal{D}(|e>, \mathbf{R}) \subseteq \mathcal{L}(\mathcal{H})$ to which truth values can be assigned via a 2-valued homomorphism and demonstrates a uniqueness theorem that allows the construction of the preferred observable.

In Bub's proposal, a *property state* is a maximal specification of the properties of the system at a particular time, defined by a Boolean homomorphism from the determined sublattice to the Boolean algebra of two elements. On the other hand, a *dynamical state* is an atom of $\mathcal{L}(\mathcal{H})$ that evolves unitarily in time following Schrödinger equation. So, dynamical states do not coincide with property states. Given a dynamical state represented by the atom $|e> \in \mathcal{L}(\mathcal{H})$, one constructs the sublattice $\mathcal{D}(|e>, \mathbf{R})$ with Kolmogorovian probabilities defined over alternative subsets of properties in the sublattice. They are the properties of the system, and the probabilities defined by $|e>$ evolve (via the evolution of $|e>$) in time. If the preferred observable is the identity operator $\mathbf{I}$, the atoms in $\mathcal{D}(|e>, \mathbf{I})$ may be pictured as a "fan" of its projectors generated by the "handle" $|e>$ (Bub, 1992, 751) or an "umbrella" with state $|e>$ again as the handle and the rays in $(|e>)^{\perp}$ as the spines. When observable $\mathbf{R} \neq \mathbf{I}$, there is a set

of handles $\{|e_{r_i}>, i = 1...k\}$ given by the nonzero projections of $|e>$ onto the eigenspaces of $\mathbf{R}$ and the spines represented by all the rays in the orthogonal complement of the subspace generated by the handles. When $dim(\mathcal{H}) > 2$, there are k 2-valued homomorphisms which map each of the handles onto 1 and the remaining atoms onto 0. The determinante sublattice (that changes with the dynamics of the system) is a partial Boolean algebra, i.e., the union of a family of Boolean algebras pasted together in such a way that the maximum and minimum elements of each one, and eventually other elements, are identified and, for every n-uple of pair-wise compatible elements, there exists a Boolean algebra in the family containing the n elements. The possibility of constructing a probability space with respect to which the Born probabilities generated by $|e>$ can be thought as measures over subsets of property states depends on the existence of sufficiently many property states defined as 2-valued homomorphisms over $\mathcal{D}(|e>, \mathbf{R})$. This is guaranteed by a uniqueness theorem that characterizes $\mathcal{D}(|e>, \mathbf{R})$ (Bub, 1997, 126). Thus constructed, the structure avoids KS-type theorems. Then, given a system S and a measuring apparatus M,

> "[...] if some quantity $\mathbf{R}$ of M is designated as always determinate, and M interacts with S via an interaction that sets up a correlation between the values of $\mathbf{R}$ and the values of some quantity $\mathbf{A}$ of S, then $\mathbf{A}$ becomes determinate in the interaction. Moreover, the quantum state can be interpreted as assigning probabilities to the different possible ways in which the set of determinate quantities can have values, where *one particular set of values represents the actual but unknown values of these quantities*." J. Bub (1992, p. 750, emphasis added)

The problem with this interpretation is that, in the case of an isolated system, there is no single element in the formalism of quantum mechanics which allows us to choose an observable, $\mathbf{R}$, rather than other. This is why the move seems fragrantly *ad hoc*. Of course, were we dealing with an apparatus, there would be a preferred observable, namely the pointer position.

Our research is focused exactly on the meaning of modality. It is thus interesting to go back to Bohm himself and recover the ideas which allowed him to develop his "causal interpretation". As noted by Bub with respect to Bohmian mechanics:

> "[...] the change in the quantum state $|\psi\rangle$ manifests itself directly at a *modal* level –the level of possibility rather than actuality– through the determinate sublattice defined by $|\psi\rangle$ and position in configuration space as the preferred determinate observable." J. Bub (1997, p. 170)

In (Bohm, 1953) Bohm proves that the probability density in his interpretation approaches the quantum probability density $|\psi|^2$. In this paper Bohm clearly recognizes the main distinction between the usual interpretation, provided by Bohr, Born, Heisenberg and Pauli, and his own causal interpretation:

> "The arbitrariness of the usual interpretation in the description of the behavior of an individual system is closely related to the assumption, already stated, that the wave function determines all physically significant properties of the system. Now, in the case of the uranium nucleus, the wave function takes the form of a packet initially entirely within the nucleus, which gradually 'leaks' through the barrier and thereafter rapidly spreads without limit in all directions. Clearly, although this wave function is supposed to describe *all* physically significant properties of the system, it cannot explain the fact that each α-particle is actually detected in a comparatively small region of space and at a fairly well-defined instant of time. *The usual interpretation states that this phenomenon must be simply accepted as an event that somehow manages to occur but in a way that is as a matter of principle forever beyond the possibility of a simultaneous and detailed 'space-time and causal description.'* Indeed, even to ask for such a description is said to be a meaningless question within the framework of the usual interpretation of the quantum theory. *In the causal interpretation, however, the postulated particles with precisely defined positions explain in a natural way why an α-particle can be detected as a fairly definite place and time, on the basis of the simple assumption that the particle existed all the time and just moved from its original location to the place where it was finally found.* Thus, even though we cannot yet observe the precise locations of our postulated particles, they already perform a real function in the theory, namely, to explain certain properties of *individual* systems which are said in the usual interpretation to be just empirically given and forever unexplainable." D. Bohm (1953, p. 464, emphasis added)

The usual interpretation follows the second demand given also by Dieks (1995), that no deterministic picture can provide a more complete description of the state of affairs. Bohm's causal interpretation, on the other hand, wishes to retain this classical feature.

> "[...] *in the usual interpretation two completely different kinds of statistics are needed.* First, there is the ordinary statistical mechanics, which treats of the distortion of systems among the quantum states, resulting from various chaotic factors such as collisions. The need of this type of statistics could in principle be avoided by means of more accurate measurements which would supply more detailed information about the quantum state, but in systems of appreciable complexity, such measurements would be impracticably difficult. Secondly, however, there is the fundamental and irreducible probability distribution, $P(x) = |\psi(x)|^2$ [...]. The need of this type of statistics cannot even in principle be avoided by means of better measurements, nor can it be explained in terms of the effects of random collision processes. [...] On the other hand, *the causal interpretation requires only one kind of probability.* For as we have seen, we can deduce the probability distribution $P(x) = |\psi(x)|^2$ as a consequence of the same random collision processes that give rise to the statistical distributions among the quantum states." D. Bohm (1953, p. 465, emphasis added)

These two statistics to which Bohm refers are the two levels of discourse present in modal interpretations (dynamical and value state). Bohm wishes to recover the classical concept of probability as lack of knowledge, however, as it will become clear in section 5, this is not achievable, in general, in the modal scheme. Possibility remains a contextual concept: a set of shared possibilities corresponding to disjoint sets of actual properties cannot be non-contextually actualized without contradictions. Classical probability can only be recovered as lack of knowledge once the definite context has been chosen.

3.4 Atomic Modal Interpretation

The atomic modal interpretation is due to G. Bacciagaluppi and M. Dickson (1997). It intends, via a factorization, to separate the state space of the system $\mathcal{H}$ in disjoint spaces $\mathcal{H}_k$. A factorization Φ of a Hilbert space $\mathcal{H}$ into a tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by an equivalence class of isomorphisms differing only by a basis transformation of the factor spaces onto themselves. It may be proved that there are many different factorizations. The question becomes now whether, by letting Φ vary, the definite properties pertaining to the different factorizations will admit a truth valuation. Bacciagaluppi has proved that this question must be answered negatively because these properties include the set of properties for which KS have shown that it is not allowed an homomorphism to the Boolean algebra **2** (Bacciagaluppi, 1995). In order to escape this no-go theorem, Bacciagaluppi and Dickson assume that there exists in nature a special set of disjoint spaces $\mathcal{H}_k$ which are the building blocks of all physical systems; i.e. a *preferred factorization* of the Hilbert space:

> "[...] we note that the idea of a preferred factorization is not, perhaps, as *ad hoc* as it might first appear. After all assuming that the universe is really made of, say, electrons, quarks, and so on, it makes good sense to take these objects to be 'real' constituents of the universe, i.e. the bearers of properties that do not supervene on the properties of subsystems." G. Bacciagaluppi and M. Dickson (1997, p. 3)

It is important to notice that in such interpretation the structure of the probability assignment becomes classical, i.e. one can define a classical joint probability distribution for any set of chosen properties. As a consequence, probability can be interpreted in terms of ignorance. In the atomic interpretation there is a single context given by the preferred factorization and thus, as in classical physics, the KS theorem does not apply. But the intrinsic modal aspect we have been referring up to now does not have to do with ignorance and, in this sense, the atomic modal interpretation may be considered closer to classical statistical interpretations of quantum mechanics (see, for example, Ballentine, 1990).

Bacciagaluppi's interpretation of modal interpretations (Bacciagaluppi, 1996), appears in an analogous fashion to Bub's proposal, closely related to Bohm's causal interpretation.

> "The properties possessed by a system in the modal interpretation are possessed *in addition* to the properties possessed by the system according to quantum mechanics. It is thus natural to call these properties 'hidden variables'. Hidden variables theories do not represent a return to classical, pre-quantum physics. Indeed, the no-go theorems for hidden variables theories show not that hidden

> variables are impossible, but that they must be in important ways different from classical physics (e.g. they are non-local). On the other hand, hidden variables theories always restore a classical way of thinking about *what there is.* In particular, the logical and probabilistic structure of a hidden variables theory is always classical: there is no 'complementarity' of hidden variables, and probabilities are rigorously Kolmogorovian. [...] The status of probabilities in the modal interpretation, however, has been the subject of some debate, I would presume partially because the original modal interpretation of van Fraassen was not intended as a theory about what there is, but, indeed, as a theory about possibilities. Thus, Dickson (1995b) has wondered whether the modal interpretation really is a no-collapse interpretation, and Healey (1995) has expressed reservations about the desirability of introducing a dynamics for the proposed properties. However, for our proposes, I would claim that, despite the name, the modal interpretation in the version of Vermaas and Dieks is a theory about actualities – albeit a stochastic one." G. Bacciagaluppi (1996, p. 74)

Bacciagaluppi is correct to point out that if one takes properties in the modal interpretation as existing in actuality, one could argue that modal interpretations are in the end, some kind of hidden variable theory. However, Bacciagaluppi does not recognize that the distinction between dynamical state and value state in van Fraassen's interpretation, or mathematical and physical state in Vermaas Dieks proposal, provides exactly this non-Kolmogorovian model regarding properties, and thus, the 'two statistics' needed in the usual interpretation, and criticized by Bohm. The mode of existence regarding the properties in the dynamical state (or in the mathematical state) is what provides a formal picture which remains non-classical, and which cannot be interpreted in terms of actuality but only as possibility, or maybe even in terms of potentiality (de Ronde, 2005, section 1.4). Bacciagaluppi and Dickson regard modal interpretations as referring to actual properties, this is why they look for a dynamical picture that governs the evolution of these properties. On the contrary, this is considered by van Fraasen and Dieks as superfluous.

4 A General Characterization of Modal Interpretations of Quantum Mechanics

Taking into account our previous analysis we are now in conditions to summarize what we consider the general characteristics of modal interpretations.

1. One of the most significant features of modal interpretations is that they stay close to the standard formulation. Following van Fraassen's recommendation, one needs to learn from the formal structure of the theory in order to develop an interpretation. This is different from many attempts which presuppose an ontology and then try to fit it into the formalism.

2. Modal interpretations are non-collapse interpretations. The evolution is always given by the Schrödinger equation of motion and the collapse of the wave function is nothing but the path from the possible to the actual, it is not considered a physical process.

3. Modal interpretations ascribe possible properties to quantum systems. The property ascription depends on the states of the systems and applies regardless of whether or not measurements are performed. There is a distinction between the level of possibility and that of actuality which are related through an interpretational rule. In addition to actual properties interpreted in the orthomodular lattice of subspaces of $\mathcal{H}$, there is a set of possible properties which may be regarded as constituting the center of the enriched lattice (see section 5).

4. Modality is not interpreted in terms of ignorance. There is no ignorance interpretation of the probability distribution assigned to the physical properties. The state of the system determines all there is to know. For modal interpretations there is no such thing as 'hidden variables' from which we could get more information. One can formulate a KS theorem for modalities which expresses the irreducible contextual character of quantum mechanics even in the case of enriching its language with a modal operator (see section 5).

It is easily seen that Born's and specially Heiseneberg's interpretations present already the basic features of modal interpretations; these ideas would be later rigourously formalized by van Fraassen's

and Kochen-Dieks' interpretations. Van Fraassen's account is *modal* because it leads, in a relatively straightforward way, to a modal logic of quantum propositions. In the Copenhagen variant, the most important point is perhaps that one should not presume this modal logic to arise from ignorance about the actual state of affairs, which is the aim of science to uncover. In other words, we do not say that a system with dynamical state φ possibly has some value state and we need to find out which one, or which one with which probability. What is important is that there are possible value states for all physical systems (i.e. possible stories about the world) that are compatible with all the observable data. Van Fraassen is closer to QL and distinguishes two kind of propositions related by a modal quantifier. However, he defines himself as an empiricist and does not interpret modality in an ontological fashion, remaining agnostic about the relation between modalities and 'the world'. K-D present a different property ascription rule given by the Schmidt decomposition. They are able to ascribe a set of definite properties to the system giving, in this way, a realistic flavor to the interpretation (see Dickson, 2002). For Dieks, one of the most pleasant features of modal interpretations is that they remain within the orthodox formulation of quantum mechanics, there is no need of hidden variables and the state of the system gives all the information there is to know. Maybe the most characteristic feature of both van Fraassen and Dieks interpretations, which go back to Heisenberg and Bohr, is the distinction between different levels of discourse; i.e. dynamical and value states in van Fraasen, and the mathematical and physical states, in Vermaas and Dieks.

The main ideas which guide the line of investigation of the Bohmian causal interpretation go against points 1, 3 and 4 of our characterization of modal interpretations. In both atomic and Bohmian variant, "possibility" is taken to express a degree of ignorance of a determined state of affairs. In Bub's proposal, ignorance about the exact *position* of the particles while, in the atomic interpretation, ignorance regarding which is the *preferred factorization* of the Universe. In both interpretations the state of system is not all there is to know, possibility is only regarded in terms of actuality and, as noticed by Bacciagaluppi (1996, p. 74), modal interpretations thus become "a theory about actualities". This is why in both atomic and Bub's interpretations, there is no distinction between different levels of discourse. That which exists, exists only at the level of actuality, exactly in the same sense as in classical statistical mechanics. In spite of the fact that one may state that this is the same empirical statement of van Fraassen, one must recall that in van Fraassen's interpretation, even though that which exists is actual (van Fraassen, 1981, section 5.3), there is a level of possibility given by the dynamical state of which he remains agnostic but which has nothing to do with ignorance.

Summarizing, on the one hand, in van Fraassen Copenhagen variant and K-D modal interpretation it remains clear that modality cannot be interpreted in terms of ignorance. However, in both interpretations the ontological significance of possibility remains untouched. The idea which van Fraassen sustains is that: *modalities are in our theories, not in the world*. Contrary to this position, one of us has proposed to interpret possibility in terms of ontological potentiality (de Ronde, 2005), taking to its last consequences the ideas of Heisenberg, and presenting possibility in terms of Aristotle, as another mode of existence, complementary to that of actuality. On the other hand, in Bub's Bohmian interpretation and also in the atomic one, modality is still thought in terms of ignorance, remaining closer to a classical statistical conception of modality. This is the character which we are specifically interested in discussing in detail. In the following section we will analyze, taking into account previous work (Domenech *et al.*, 2006a; Domenech *et al.*, 2006b) whether this idea can be consistently maintained in the orthodox formalism of quantum mechanics.

5 The Contextual Character of Modal Interpretations of Quantum Mechanics

In order to stay away from inconsistencies when speaking about properties which pertain to the system, one must acknowledge the limitations imposed by the KS theorem. To do so, modal interpretations assign to the system only a set of definite properties. However, it has been shown by Bacciagaluppi (1995), R. Clifton (1995, 1996) and later by Vermaas (1997, 1999b) that this is not achievable when talking about properties which pertain to different contexts (see also Bacciagaluppi and Vermaas, 1999).

At first sight it might seem paradoxical that, even though quantum mechanics talks about modalities, KS theorem refers to actual values of physical properties. Elsewhere, and following the line of thought of quantum logic, we have investigated the question whether KS theorem has something to say about possibility and its relation to actuality (Domenech *et al.*, 2006a; Domenech *et al.*, 2006b). The answer

was provided via a characterization of the relations between actual and possible properties pertaining to different contexts. By applying algebraic and topological tools we studied the structure of the orthomodular lattice of actual propositions enriched with modal propositions. Let us briefly recall the results. As usual, given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it (and about a set of propositions that are compatible with it) and predicate probabilities about the other ones. This is to say that one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a Boolean algebra. In rigorous terms, for each proposition P, if we refer with $\Diamond P$ to the possibility of P, then $\Diamond P$ will be a central element, i.e., a complemented element which satisfies the distributive law for every set of three elements of the lattice. The orthomodular lattice thus expanded includes propositions about possibility. If P is a proposition about the system and P occurs, then it is trivially possible that P occurs. This is expressed as $P \leq \Diamond P$. If we identify P with the value-attribution proposition $< \mathbf{A}, \sigma >$ as defined by van Fraassen, we may say that the classical consequences of P coincide with those of its correspondent state-attribution proposition $[\mathbf{A}, \sigma]$. In fact, to assume an actual property and a complete set of properties that are compatible with it determines a context in which the classical discourse holds. Classical consequences that are compatible with it, for example probability assignments to the actuality of other propositions, shear the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible property. This is interpreted as, if P is a property of the system, $\Diamond P$ is the smallest central element greater than P. With these tools, we are able to give an extension of the orthomodular structure by adding a possibility operator that fulfills the mentioned requirements. More precisely, the extension is a class of algebras, called Boolean saturated orthomodular lattices, that admits the orthomodular structure as a reduct and we demonstrate that they are a variety, i.e., definable by equations. Complete orthomodular lattices are examples of them (Domenech *et al*, 2006a). We have also found its logic (Domenech *et al.*, 2007b). This algebraic construction is what allows us to consistently expand the structure of actual properties to include modal properties. This is so because we have proved that every orthomodular lattice may be embedded in a Boolean saturated orthomodular one.

If $\mathcal{L}$ is an orthomodular lattice and $\mathcal{L}^\Diamond$ a Boolean saturated orthomodular one such that $\mathcal{L}$ can be embedded in $\mathcal{L}^\Diamond$, we say that $\mathcal{L}^\Diamond$ is a modal extension of $\mathcal{L}$. Given $\mathcal{L}$ and a modal extension $\mathcal{L}^\Diamond$, we define the *possibility space* as the subalgebra of $\mathcal{L}^\Diamond$ generated by $\{\Diamond P : P \in \mathcal{L}\}$. We denote by $\Diamond\mathcal{L}$ this space and it may be proved that it is a Boolean subalgebra of the modal extension. The possibility space represents the modal content added to the discourse about properties of the system.

Within this frame, the actualization of a possible property acquires a rigorous meaning. Let $\mathcal{L}$ be an orthomodular lattice, $(W_i)_{i\in I}$ the family of Boolean sublattices of $\mathcal{L}$ and $\mathcal{L}^\Diamond$ a modal extension of $\mathcal{L}$. If $f : \Diamond\mathcal{L} \to \mathbf{2}$ is a Boolean homomorphism, an actualization compatible with f is a global valuation $(v_i : W_i \to \mathbf{2})_{i\in I}$ such that $v_i \mid W_i \cap \Diamond\mathcal{L} = f \mid W_i \cap \Diamond\mathcal{L}$ for each $i \in I$.

A kind of converse of this possibility of actualizing properties may be read as an algebraic representation of the Born rule, something that has no place in the orthomodular lattice alone (Domenech *et al.*, 2006a):

Theorem 5.1 *Let $\mathcal{L}$ be an orthomodular lattice, W a Boolean sublattice of $\mathcal{L}$ and $f : W \to \mathbf{2}$ a Boolean homomorphism. If we consider a modal extension $\mathcal{L}^\Diamond$ of $\mathcal{L}$ then there exists a Boolean homomorphism $f^* : \langle W \cup \Diamond\mathcal{L}\rangle_{\mathcal{L}^\Diamond} \to \mathbf{2}$ such that $f^* \mid W = f$.*

Compatible actualizations represent the passage from possibility to actuality, they may be regarded as formal constrains when applying the interpretational rule proposed by Dieks. When taking into account compatible actualizations from different contexts, the following KS theorem for modalities can be proved (Domenech *et al.*, 2006a):

Theorem 5.2 *Let $\mathcal{L}$ be an orthomodular lattice. Then $\mathcal{L}$ admits a global valuation iff for each possibility space there exists a Boolean homomorphism $f : \Diamond\mathcal{L} \to \mathbf{2}$ that admits a compatible actualization.* □

This theorem shows that no enrichment of the orthomodular lattice with modal propositions allows to circumvent the contextual character of the quantum language. This is why we have called 5.2 the modal Kochen-Specker (MKS) theorem.

This may also be seen from a topological point of view. Let us consider $\mathcal{L}^\Diamond$, the modal extension of $\mathcal{L}$. Then, the spectral sheaf p defined in section 2 is a subsheaf of $p_{\mathcal{L}^\Diamond}$. In this case we refer to $p_{\mathcal{L}^\Diamond}$ as a *modal extension* of p. It is clear that local sections of p can be seen as local sections of $p_{\mathcal{L}^\Diamond}$. Then

we define the set $Sec(\Diamond\mathcal{L}) = \{\nu : (\Diamond\mathcal{L}] \rightarrow E_{\mathcal{L}^\diamond} : \nu$ is principal section of $p_{L^\diamond}\}$. Since $\Diamond\mathcal{L}$ is a Boolean algebra, it is a subdirect product of **2**. Thus, it always exists a Boolean homomorphism $f : \Diamond\mathcal{L} \rightarrow 2$, resulting $Sec(\Diamond\mathcal{L}) \neq \emptyset$. From a physical point of view, $Sec(\Diamond\mathcal{L})$ represents all physical properties as possible properties. The fact that $Sec(\Diamond\mathcal{L}) \neq \emptyset$ shows that, *in the frame of possibility*, one may talk simultaneously about all physical properties. The (always possible) choice of a context in which any possible property pertaining to this context can be considered as an actual one, may be formalized in the following way: let $\mathcal{L}$ be an orthomodular lattice, W a Boolean sublattice of $\mathcal{L}$, $q \in W$ and $\mathcal{L}^\Diamond$ be a modal extension of $\mathcal{L}$. If $\nu \in Sec(\Diamond\mathcal{L})$ such that $\nu(\Diamond q) = 1$ then an actualization of q compatible with ν is an extension $\nu' : U \rightarrow E_{\mathcal{L}^\diamond}$ such that $(\langle W \cup \Diamond\mathcal{L}\rangle_{L^\diamond}] \in U$. Then we may prove (Domenech *et al.*, 2006b) that, if $\nu \in Sec(\Diamond\mathcal{L})$ such that $\nu(\Diamond q) = 1$, then there exists an actualization of q compatible with ν. It is also possible to represent the Born rule in terms of continuous local sections of sheaves: let $\mathcal{L}$ be an orthomodular lattice, W a Boolean sublattice of $\mathcal{L}$, and $\nu : (W] \rightarrow E_{\mathcal{L}}$ a principal local section. If we consider a modal extension $\mathcal{L}^\Diamond$ of $\mathcal{L}$ then there exists an extension $\nu' : U \rightarrow E_{\mathcal{L}^\diamond}$ such that $\langle W \cup \Diamond\mathcal{L}\rangle_{\mathcal{L}^\diamond} \in U$. This rule quantifies possibilities from a chosen spectral algebra and is a kind of the converse of the possibility of actualizing properties to which we have referred before. Now, an actualization compatible with ν is a global section $\tau : \mathcal{W}_{\mathcal{L}} \rightarrow E_{\mathcal{L}}$ of $p_{\mathcal{L}}$ such that $\tau(W \cap \Diamond\mathcal{L}) = \nu(W \cap \Diamond\mathcal{L})$. Thus we may state the following theorem, the topological version of the MKS theorem:

Theorem 5.3 *Let $\mathcal{L}$ be an orthomodular lattice. Then $p_{\mathcal{L}}$ is a global section τ iff for each modal extension $\mathcal{L}^\Diamond$ there exists $\nu \in Sec(\Diamond\mathcal{L})$ such that τ is a compatible actualization of ν.* □

In view of this theorem, since any global section of the spectral sheaf is a compatible actualization of a local one belonging to $Sec(\Diamond\mathcal{L})$, a global actualization that would correspond to a family of compatible valuations is prohibited. Thus, the theorem provides the same conclusion of the MKS theorem, but now from a topological point of view.

The scope of the MKS theorem and its topological version 5.3 supersedes the usual KS-type theorems, which only refer to the actual values of physical properties. It allows to take into account also possible properties which enlarge the expressive power of the discourse. In spite of the fact that at first sight it may be thought that referring to possibility could help to circumvent contextuality, allowing to refer to physical properties belonging to the system in an objective way that resembles the classical picture, our theorems show that this cannot be consistently maintained.

Discussion

Our interest in KS-type theorems in relation to modalities steams from the fact that quantum mechanics has referred, from its very beginning, not only to actuality but also to possibility. In this sense, our problem has been to clarify the relation between contextuality and modality in quantum mechanics. We have chosen a logical approach because, though the main contribution of a logical calculus is actually rather technical, it makes visible the structure in which propositions lie and provides bounds to a consistent discourse. It is a powerful tool when dealing with non-classical propositions.

We have shown that the addition of modalities to the discourse about the properties of a quantum system genuinely enlarges its expressive power. More precisely, the usual orthomodular propositional structure $\mathcal{L}$ that does not contain modal elements is embedded in a Boolean saturated orthomodular lattice $\mathcal{L}^\Diamond$, its modal extension, to obtain a common frame. But in view of MKS theorem, a global actualization that would correspond to a family of compatible valuations is prohibited. Thus, contextuality remains a central feature of quantum systems even when possibilities are taken into account by enriching the structure with modal propositions.

The MKS theorem imposes clear constrains in order to put forward an interpretation in terms of objective probabilities, such as those proposed by Popper in terms of propensities. If one is willing to have an interpretation of probability in terms of objective chance, i.e. a certain measure over possible worlds, there must be some realistic support, a sense in which these possible worlds are real. As noted by Schrödinger, "*A probabilistic assertion presupposes the full reality of its subject.*" Our modal extension provides exactly the adequate framework to think in terms of possible worlds, but at the same time, through our MKS theorem, it precludes the interpretation of possibility in terms of objective probability.

Finally, we would like to remark that our formalism also provides a formal meaning in an algebraic frame to the Born rule, something that has been discussed recently by D. Dieks (2007) in relation to the possible derivation of a preferred probability measure.

Modal interpretations still have to present a consistent image of the theory. In words of van Fraassen, the theory should tell us how the world is like if the theory is true. We believe that in order to do so, a central feature that needs to be further investigated is the interpretation of modalities in quantum mechanics. Regarding the idea of introducing potentiality in quantum mechanics (see, for example, Karakostas, 2004 and Smets, 2005), it remains for us of great interest to study the constrains under which such interpretation may be applied. It is important to stress that the idea of interpreting modalities in terms of ontological potentiality (de Ronde, 2005) may allow to escape the constrains imposed by the MKS theorem, as will be discussed in a forthcoming paper (Domenech *et al.*, 2007a).

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